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The Asymmetric Contact Process at Its Second Critical Value

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The asymmetric contact process on $\mathbf{Z}$ has two distinct critical values $\lambda_1 < \lambda_2$ (at least with sufficient asymmetry). One can consider the process on $\{0,..., N\}$ and analyze the time (which we call σ_N) till complete vacancy starting from complete occupation. Its behavior has already been resolved for all regions of λ except for $\lambda = \lambda_2$. For this value, Schinazi proved that $\lim_{N \to \infty} \log \sigma_N / \log N = 2$ in probability and conjectured that σ_N / N^2 converges in distribution. It is that result that we prove in this paper. We rely heavily on the Brownian motion behavior of the edge particle, which comes from Galves and Presutti and Kuczek.

KEY WORDS: Asymmetric contact process; edge speeds.

1. INTRODUCTION

The asymmetric contact process on $\mathbf{Z}$ is a continuous-time Markov process ξ_t on $\{0, 1\}^{\mathbf{Z}}$ that evolves with the following rates: if $\xi_t(x) = 1$, then $\xi_t(x)$ changes to 0 at rate 1; if $\xi_t(x) = 0$ then $\xi_t(x)$ changes to 1 at rate $\lambda_l \xi_t(x+1) + \lambda_r \xi_t(x-1)$. Casually speaking, we can describe it as follows: Every integer is either infected or healthy. If $x \in \mathbf{Z}$ is infected, then the infection dies out (i.e., x becomes healthy) at rate 1; if there is an infection at $y \in \mathbf{Z}$, then it infects $y-1$ at rate λ_l and infects $y+1$ at rate λ_r. We let $\lambda_l = \theta\lambda$ and $\lambda_r = (2-\theta)\lambda$, where $\theta \in [0, 2]$ is fixed and $\lambda > 0$ is the parameter. In this formulation, $\theta = 1$ gives us the basic contact process, which has been studied for two decades. For background material, see Liggett[8] or Durrett.[2]

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The nature of phase transitions in the asymmetric case can be more interesting (and less clear) than in the symmetric case. Let ξ_t^0 be the asymmetric contact process whose initial state is $\xi_0^0(x) = 1$ iff $x = 0$. Define the following two critical values:

$$\lambda_1 = \sup\left\{\lambda : P\left(\forall t > 0, \sum_{x \in \mathbf{Z}} \xi_t^0(x) \geqslant 1\right) = 0\right\}$$

and

$$\lambda_2 = \sup\left\{\lambda : \limsup_{t \to \infty} P(\xi_t^0(0) = 1) = 0\right\}$$

The first critical value tells us when there is global survival and the second critical value tells us when there is local survival; it is clear that $\lambda_1 \leqslant \lambda_2$. While the two critical values are equal in the basic contact process, it is known that they are not equal when the asymmetry is sufficiently great—that is, when θ is sufficiently far from 1.[10] While it is certainly not unreasonable to expect that they are different for any θ other than one—Schonmann conjectured exactly that—nothing more is known.

In discussing the critical values and phase transitions, it is useful to look at edge speeds. Let $\xi_t^{(-\infty,0]}$ be the asymmetric contact process whose initial state is $\xi_0^{(-\infty,0]}(x) = 1$ iff $x \leqslant 0$. Then let r_t be the rightmost particle:

$$r_t = \sup\{x \in \mathbf{Z} \mid \xi_t^{(-\infty,0]}(x) = 1\}$$

It is known that there exists $\alpha_r \in [-\infty, \infty)$ such that $\lim_{t \to \infty} r_t/t = \alpha_r$ a.s. Similarly, using the leftmost particle l_t, there exists $\alpha_l \in [-\infty, \infty)$ such that $\lim_{t \to \infty} l_t/t = -\alpha_l$ a.s.

In Schonmann,[10] the critical values are related directly to the edge speeds via the following results:

$$\begin{aligned} \lambda = \lambda_1 &\Leftrightarrow \alpha_l + \alpha_r = 0 \\ \lambda = \lambda_2 &\Leftrightarrow \min(\alpha_l, \alpha_r) = 0 \end{aligned} \tag{1.1}$$

For this paper, we will only concern ourselves with θ such that $\lambda_1 < \lambda_2$. From the above characterizations we know that one edge speed is strictly greater than the other throughout the intermediate region. Without loss of generality, we will assume that $\alpha_r > \alpha_l$ on the region $[\lambda_1, \lambda_2]$. We point out that we cannot say if $\alpha_r > \alpha_l$ is the same as $\theta < 1$, though it certainly would not be unreasonable to think that it is. In fact, we only know $\alpha_r \geqslant \alpha_l$ for θ sufficiently close to 0. This matter is, however, only a side point for the purposes of this paper.

We now wish to relate the phase transitions of the asymmetric contact process to its behavior when restricted to a finite set. Fix a (large) integer N and denote by η_t the asymmetric contact process on $\{0, 1,..., N-1\}$ with the births on -1 and N suppressed. Then let σ_N be the first time that η_t has no infected particles, starting at the initial configuration $\eta_0(x)=1$, $\forall x \in [0, N-1]$. The order of magnitude of the extinction time σ_N can change dramatically when λ moves on and off its critical values.

In the subcritical basic contact process, Durrett and Liu[3] showed that the extinction time behaves logarithmically in the sense that $\sigma_N/\log N$ converges in probability to a positive finite constant. Schinazi[9] pointed out, while beginning the analysis of σ_N in the asymmetric case, that their argument can be adapted to the asymmetric case. Durrett and Schonmann[10] gave a proof for the supercritical basic contact process that the extinction time behaves exponentially in the sense that $\log \sigma_N/N$ converges in probability to a positive finite constant. This, too, can be adapted to the asymmetric case. For the critical basic contact process, the extinction time behaves like a polynomial in the sense that $\sigma_N/N \xrightarrow{P} \infty$ while $\sigma_N/N^4 \xrightarrow{P} 0$.[5]

Schinazi[9] showed that the extinction time for $\lambda \in [\lambda_1, \lambda_2)$ behaves like N in the sense that $\sigma_N/N \xrightarrow{P} -1/\alpha_l$. This leaves only the behavior of σ_N when $\lambda=\lambda_2$. For that case, Schinazi[9] proved that σ_N behaves like N^2 in the weaker sense that $\log \sigma_N/\log N^2 \xrightarrow{P} 1$. He then conjectured that σ_N/N^2 would converge in distribution. It is this result that will be proved here. Specifically, the following result is shown.

Theorem 1.1. Assume that θ is such that $\lambda_1 < \lambda_2$. Then at $\lambda = \lambda_2$ we have that

$$\lim_{N\to\infty} \frac{\sigma_N}{N^2} = \inf\{t\colon |B_t| = 1\} \qquad \text{in distribution}$$

where B is a Brownian motion with some nontrivial diffusion constant.

From now on all discussion is focused on the second critical value, where, using (1.1),

$$\alpha_l = 0 \qquad \text{and} \qquad \alpha_r > 0 \tag{1.2}$$

To analyze the behavior of σ_N, we must first analyze the behavior of the edges of the finite process, which we will denote by $l_t^{\eta,[0,N-1]}$ and $r_t^{\eta,[0,N-1]}$. Actually, of the two directions, the left—which has speed 0 while the other speed is positive—should seem to be more significant. Indeed, if we could show that our left edge has, in some sense, Brownian fluctuations, then the convergence in distribution of σ_N/N^2 could become trivial. This will now be made more precise.

In the supercritical basic contact process, the leftmost particle converges to Brownian motion when scaled in the usual way. This was first shown by Galves and Presutti.[6] Kuczek[7] has a rather intuitive proof in the supercritical oriented percolation setting which can be adapted without great pain to our setting. So, this result is stated as Theorem 1.2 below.

Theorem 1.2. In the Skorohod topology,

$$\lim_{N\to\infty} l_{N^2t}/N = B(t) \qquad \text{in distribution}$$

given $\{0 \text{ survives}\}$, where B is a Brownian motion with some nontrivial diffusion constant.

The process B in Theorem 1.1 is the same process as that in Theorem 1.2. While the diffusion constant for this Brownian motion may not be 1, this is of no interest for the purposes of the main result. Whenever we refer to Brownian motion, it will have the same diffusion constant as the one in the above theorem.

While the Brownian motion behavior of l_t is known, we actually seek information about $l_t^{\eta, [0, N-1]}$. Of course, it cannot really fluctuate like Brownian motion, since it always lies in $[0, N]$. However, the bound of N is merely a technicality which should not bother us; after all,

$$\sigma_N = \inf\{t > 0 : l_t^{\eta, [0, N-1]} \geqslant N\}$$

As for the bound of 0, it can lead us to believe that $l_t^{\eta, [0, N-1]}$ should converge to reflecting Brownian motion when appropriately scaled.

For reasons mentioned in the previous paragraph, we define

$$L_t = \begin{cases} l_t^{\eta, [0, N-1]} & \text{if} \quad t < \sigma_N \\ |N + \bar{B}_{t-\sigma_N}| & \text{if} \quad t \geqslant \sigma_N \end{cases}$$

where $\bar{B}_s$ is a Brownian motion independent of the contact process. For our purposes, L_t is the same as $l_t^{\eta, [0, N-1]}$; however, the change allows us conveniently to state the following theorem, which we will later prove.

Theorem 1.3. Assume that $\lambda = \lambda_2$ with $\alpha_l = 0$ and $\alpha_r > 0$. Let B be a Brownian motion (with a diffusion constant as in Theorem 1.2). Then

$$\lim_{N\to\infty} L_{N^2t}/N = |B(t)| \qquad \text{in distribution}$$

From this theorem, we can immediately conclude Theorem 1.1. The next section contains definitions and a brief outline of the remaining sections, which contain the proofs.

2. DEFINITIONS

Some time must now be taken to recall the graphical construction of the asymmetric contact process and to discuss some related notation that will be used extensively. For all $x \in \mathbf{Z}$ let $\{I_l^x(n): n \geqslant 1\}$, $\{I_r^x(n): n \geqslant 1\}$, and $\{D^x(n): n \geqslant 1\}$ be independent Poisson processes at rates λ_l, λ_r and 1, respectively. For each $D^x(n)$, mark a δ over the point $(x, D^x(n))$; for each $I_l^x(n)$ [resp. $I_r^x(n)$], draw an arrow from $(x, I_l^x(n))$ to $(x-1, I_l^x(n))$ [resp. from $(x, I_r^x(n))$ to $(x+1, I_r^x(n))$]. Given (x, s) and (y, t), we say that there is a path from (x, s) to (y, t) (or a path from x at time s to y at time t) if there exist integers $y_0 < y_1 < \cdots < y_m$ and positive reals $t_0 < t_1 < \cdots < t_{m+1}$ such that $y_0 = x$, $t_0 = s$, $y_m = y$, $t_{m+1} = t$, there is an arrow from (y_{i-1}, t_i) to (y_i, t_i) (for each $0 < i \leqslant m$), and there is no δ marked over any point in $\{y_i\} \times [t_i, t_{i+1}]$ (for $0 \leqslant i \leqslant m$). Given such integers and positive reals, define a path to be the set consisting of the points in $[y_{i-1}, y_i] \times t_i$ (for $0 < i \leqslant m$) and the points in $\{y_i\} \times [t_i, t_{i+1}]$ (for $0 \leqslant i \leqslant m$). For more generality, we may replace x and y above by sets of integers (or s and t by sets of positive reals) with the obvious meaning. Also, we write $A_1 \times U_1 \leftrightarrow A_2 \times U_2$ as shorthand for "there is a path from $A_1 \times U_1$ to $A_2 \times U_2$." Now, given $A \subset \mathbf{Z}$ and given $s \in \mathbf{R}^+$, define (for reals $t > s$ and integers x)

$$\xi_t^{A \times s}(x) = 1_{A \times s \leftrightarrow (x, t)}$$

This yields a version of the asymmetric contact process as defined earlier. Now define (for reals $t > s$)

$$l_t^{A \times s} = \inf\{x: \xi_t^{A \times s}(x) = 1\}$$

as well as

$$r_t^{A \times s} = \sup\{x: \xi_t^{A \times s}(x) = 1\}$$

with the usual convention that the infimum of the empty set is infinity. Next, let N be a positive integer. For $0 \leqslant x \leqslant N-1$, define

$$\eta_t^{A \times s}(x) = 1_{A \times s \leftrightarrow (x, t) \text{ inside } [0, N-1] \times [0, \infty)}$$

where "inside" simply means "and there is such a path lying inside"; that is, $\eta_t^{A \times s}(x) = 1$ iff there is a path from $A \times s$ to (x, t) lying inside $[0, N-1] \times [0, \infty)$. This yields a version of the asymmetric process on $[0, N-1]$. For the left and right edges of this process, define

$$l_t^{\eta, A \times s} = \min\{N, \inf\{0 \leqslant x \leqslant N-1: \eta_t^{A \times s}(x) = 1\}\}$$

as well as

$$r_t^{\eta, A\times s} = \max\{0, \sup\{0 \leqslant x \leqslant N-1 : \eta_t^{A\times s}(x) = 1\}\}$$

In the above definitions, if $s = 0$, we suppress s and if $A = \{0\}$, we suppress A.

We will say that η^A dies out at time t if $\sum_{x\in \mathbf{Z}} \eta_s^A(x)$ is positive for $0 < s < t$ and is zero for $s = t$ and we say that η^A is alive at time t if η^A has not died out before (or at) time t. Using this terminology, let σ_N represent the time that $\eta^{[0, N-1]}$ dies out.

Having completed our discussion on the graphical representation and its associated definitions, we can now begin to focus on Theorem 1.3. Rather than trying to prove the theorem from scratch, we would like to make use of Theorem 1.2. Unfortunately, l^η does not look like l; we immediately have a problem with the boundary. So we seek to create a process that is similar to l^η, but does not immediately hit the boundary.

To do this, consider a process on $[0, N]$ starting with occupations on $[\varepsilon N, N)$. In this case, the boundary is not an immediate issue. Of course, the left edge in this process will eventually hit its extreme. Let us say it hits 0 rather than N. Then we start to have the same problem we had to begin with. So at that moment modify the configuration by forcing a vacancy at every site in $[0, \varepsilon N)$. Once again, the boundary is removed from immediate concern. Proceed in this manner until the edge hits N rather than 0. With this thinking in mind, let $\varepsilon > 0$ and make the following recursive definitions:

$$T^0 = 0$$

$$H^1 = [\varepsilon N, N)$$

$$X^1(t) = \frac{1}{N} l_{N^2 t}^{\eta, [\varepsilon N, N)}$$

$$T^1 = \inf\{t \mid X^1(t) \notin (0, 1)\}$$

If $T^m < \infty$ and $X^m(T^m) = 0$, then

$$H^{m+1} = \eta_{T^m}^{H^m \times N^2 T^{m-1}} \cap [\varepsilon N, N)$$

$$X^{m+1}(t) = \frac{1}{N} l_{N^2(t+T^m)}^{\eta, H^{m+1} \times N^2 T^m}$$

$$T^{m+1} = T^m + \inf\{t \mid X^{m+1}(t) \notin (0, 1)\}$$

Otherwise (and this is for convenience only)

$$H^{m+1} = (-\infty, \infty)$$
$$X^{m+1} = 1 + B$$

where B is a Brownian motion, and

$$T^{m+1} = \infty$$

Then concatenate the processes together by defining

$$Y^{\varepsilon}(t) = \sum_{m=1}^{\infty} X^{m}(t - T^{m-1})\, 1_{[T^{m-1}, T^{m})}$$

Y^{ε} is defined so that its behavior can be compared to that of the left edge on the line (as opposed to the interval), whose behavior is understood. We will prove a result for Y^{ε} based upon Theorem 1.2. To state the result, let $\{B^{i}\}$ be independent Brownian motions. Then make the following recursive definitions:

$$X^{1}_{\infty} = \varepsilon + B^{1}$$
$$T^{1}_{\infty} = \inf\{t \mid X^{1}_{\infty} \notin (0, 1)\}$$

If $X^{m}_{\infty}(T^{m}_{\infty}) = 0$, then

$$X^{m+1}_{\infty} = \varepsilon + B^{m+1}$$
$$T^{m+1}_{\infty} = T^{m}_{\infty} + \inf\{t \mid X^{m+1}_{\infty} \notin (0, 1)\}$$

Otherwise

$$X^{m+1}_{\infty} = 1 + B^{m+1}$$
$$T^{m+1}_{\infty} = \infty$$

Finally, define

$$Y^{\varepsilon}_{\infty}(t) = \sum_{m=1}^{\infty} X^{m}_{\infty}(t - T^{m-1}_{\infty})\, 1_{[T^{m-1}_{\infty}, T^{m}_{\infty})}$$

With these definitions, we can state the proposition to which we have alluded.

Proposition 2.1. In the Skorohod topology,

$$\lim_{N \to \infty} Y^{\varepsilon}_{N} = Y^{\varepsilon}_{\infty} \qquad \text{in distribution}$$

This proposition is the true-key to both Theorem 1.3 and Theorem 1.1. To obtain Theorem 1.3 from Proposition 2.1, some technical care must be taken in proving an appropriate relationship between L and Y^{η}_N. However, after understanding Y^{η}_N through the proof of Proposition 2.1, this detail becomes simply tedious, but not difficult. We will avoid this detail and focus solely on the real issue: Proposition 2.1. (We should add here that we do not even require Theorem 1.3 and could go quickly to Theorem 1.1. However, the statement of Theorem 1.3 gives a better understanding of the system than does the statement of Proposition 2.1.)

In the next section, we prove an intermediate result which brings the Brownian motion behavior into the finite state space. Then in Section 4 we will complete the proof of Proposition 2.1.

3. BEHAVIOR OF THE LEFT EDGE ON $[0, N-1]$

For this section, let G_N be a random subset of $[\varepsilon N, \varepsilon N+\sqrt{N}]$, independent of ξ, and let

$$A_N = \{ |G_N| \geqslant \log N \}$$

$$S_N = \inf\{ t \mid l^{\eta_N, G_N}_{N^2 t} \notin (0, N) \}$$

$$Z_N(t) = \begin{cases} l^{\eta_N, G_N}_{N^2 t}/N & \text{if} \quad t \leqslant S_N \\ l^{G_N}_{N^2 t}/N & \text{if} \quad t > S_N \end{cases}$$

[The definition of $Z_N(t)$ for $t > S_N$ is for convenience only.] In this section we shall prove the following proposition; we will make use of it in the next section.

Proposition 3.1. Z_N on A_N converges in distribution to $\varepsilon + B$, where B is a Brownian motion, if $\liminf_N P(A_N) > 0$.

The condition on A_N is a trivial technicality; indeed, if the probabilities of those events are not large, then the result is not very helpful anyway. As we will see in the next section, the probability of those events will be quite high when we apply this proposition with $G_N = H^i_N \cap [\varepsilon N, \varepsilon N + \sqrt{N}]$.

This proposition follows from the following two lemmas using a common convergence result (see Theorem 4.1 of Billingsley[11]). The first lemma is a basic extension of Theorem 1.2 and the second is the link between the finite state space and the infinite one. (We should note that the Skorohod topological space is also a separable metric space.)

Lemma 3.2. $l^{G_N}_{N^2 t}/N$ on A_N converges in distribution to $\varepsilon + B$, where B is a Brownian motion.

Lemma 3.3. $Z_N(t) - l^{G_N}_{N^2 t}/N$ converges in probability to 0 (in the Skorohod space).

Proof of Lemma 3.2. First, define $R_N = \inf\{s \mid (l^{G_N}_s, s) \text{ survives}\}$, and

$$\tilde{A}_N = A_N \cap \{R_N < \sqrt{N}\} \cap \{l^{G_N}_{R_N} \in [\varepsilon N - N^{2/3}, \varepsilon N + N^{2/3}]\}$$

Now, it follows (from Theorems 12 and 13 of Schonmann[10]) that

$$P(R_N < \sqrt{N} \mid A_N) \to 1$$

Further, letting

$$C_N = \{\varepsilon N + N^{2/3} > r_t^{(-\infty, \varepsilon N + \sqrt{N}]} \geqslant l_t^{[\varepsilon N, \infty)} > \varepsilon N - N^{2/3}, \forall t < \sqrt{N}\}$$

it follows from a now-standard argument for edge speeds that

$$P(C_N) \to 1$$

Letting C_N^c denote the complement of C_N, it then follows that

$$\begin{aligned}
\lim_N P(\tilde{A}_N \mid A_N) &\geqslant \lim_N P(\{R_N < \sqrt{N}\} \cap C_N \mid A_N) \\
&\geqslant \lim_N \frac{P(A_N \cap \{R_N < \sqrt{N}\}) - P(C_N^c)}{P(A_N)} \\
&= \lim_N P(R_N < \sqrt{N} \mid A_N) \\
&= 1
\end{aligned}$$

So it suffices to show the lemma for $\tilde{A}_N$ instead of for A_N.

To deal with the finite dimensional distributions, we pick $L > 0$ and will prove the lemma when considering the processes as functions over $[N_0^{-3/2}, L]$ for each N_0. Let $N > N_0$. Then, for each $t \in [N_0^{-3/2}, L]$, it is clear that $t > R_N/N^2$ on $\tilde{A}_N$. Also, for each $s > R_N$, the following equality holds on $\tilde{A}_N$:

$$l_s^{G_N} = l_s^{l^{G_N}_{R_N} \times R_N}$$

On $\tilde{A}_N$ these two facts imply that

$$\frac{l^{G_N}_{N^2 t}}{N} \stackrel{d}{=} \frac{l^{G_N}_{R_N}}{N} + \frac{\check{l}_{N^2(t - R_N/N^2)}}{N} \tag{3.1}$$

as functions over $[N_0^{-3/2}, L]$ and where $\check{l}$ is an independent copy of l on its copy of $\{0 \text{ survives}\}$. Now, since $l^{G_N}_{R_N} \in [\varepsilon N - N^{2/3}, \varepsilon N + N^{2/3}]$, the first addend of the right-hand side of (3.1) converges in distribution to ε. On the other hand, Theorem 1.2 says that the second addend converges in distribution to a Brownian motion. This is sufficient to show that the finite dimensional distributions of $l^{G_N}_{N^2t}/N$ (on $\tilde{A}_N$) converge to $\varepsilon + B$. As for the tightness of $l^{G_N}_{N^2t}/N$ (on $\tilde{A}_N$), it follows using the ideas of Kuczek,[7] just as it did in Theorem 1.2.

Proof of Lemma 3.3. First of all, it suffices to prove the result in the L^∞ metric space, as the Skorohod topology can be generated via a metric that is dominated by the L^∞ norm. That is, we will show

$$\sup_t |Z_N(t) - l^{G_N}_{N^2t}/N| \xrightarrow{P} 0 \tag{3.2}$$

That said, let us now make the following four observations:

- If $l^{G_N}_t$ has yet to hit 0 or N at time N^2s, and if η^{G_N} is still alive at time N^2s, then $Z_N(t) = l^{G_N}_{N^2t}/N$ for $t \leqslant s$.
- If N^2s is the first time that $l^{G_N}_t = 0$, and if η^{G_N} is still alive at time N^2s, then $Z_N(t) = l^{G_N}_{N^2t}/N$ for all t. (Given those conditions, we have that $l^{\eta, G_N}_{N^2t} = l^{G_N}_{N^2t}$ for all $t \leqslant s$. The observation follows.)
- If N^2s is the first time that $l^{G_N}_t = N$, and if η^{G_N} had not died out before time N^2s, then $Z_N(t) = l^{G_N}_{N^2t}/N$ for all t. (In this case, η^{G_N} must die out at time N^2s.)
- If η^{G_N} dies out at time N^2s, then $Z_N(t) = l^{G_N}_{N^2t}/N$ for all $t > s$.

It follows from these four remarks that $Z_N(t) = l^{G_N}_{N^2t}/N$ at all times t except (possibly) at a single time (namely, the time when η^{G_N} dies out).

Hence, given $\beta > 0$,

$$P(\exists t, l^{G_N}_{N^2t}/N - Z_N(t) \geqslant \beta) \leqslant P(\exists t, l^{G_N}_{N^2t}/N - \lim_{v \nearrow t} l^{G_N}_{N^2v}/N \geqslant \beta) \tag{3.3}$$

But, taking advantage of the previous lemma, we have that the right-hand side of (3.3) converges (as N goes to ∞) to 0 (as Brownian motion has, of course, no discontinuities).

Now we must deal with the case that $Z_N(v) - l^{G_N}_{N^2v}/N > \beta$ for some v. That would mean (i) η^{G_N} must die out at time N^2S_N, (ii) $l^{G_N}_{N^2S_N} < (1-\beta)N$, and (iii) $l^{G_N}_{N^2t} > 0$ for each $t \leqslant S_N$. Further, for that to be true, any path from $G_N \times 0$ to $(l^{G_N}_{N^2S_N}, N^2S_N)$ must hit $N \times (0, \infty)$ and no such path can ever hit $0 \times (0, \infty)$. Otherwise η^{G_N} would still be alive at time v. Define $\tau_1 = \inf\{t > 0 : \xi^{G_N}_t(N) = 1\}$ and, for each i,

$$\tau_{i+1} = \inf\{t > \tau_i : \xi^{G_N}_t(N) = 1, \lim_{s \nearrow t} \xi^{G_N}_s(N) = 0\}$$

So if $Z_N(v) - l^{G_N}_{N^2t}/N > \beta$, then the following event must hold for some i:

$$Q_i = \{\exists N^2 t \in (\tau_i, N^2 S_N], \exists x \in (0, 1-\beta)N], (N, \tau_i) \leftrightarrow (x, N^2 t), \eta^{G_N}_{N^2t}(x) = 0\}$$

Now, it is easy to see that $\lim_{N \to \infty} P(N^2 S_N < N^3 < \tau_{N^4}) = 1$. So the proof is complete after establishing

$$\lim_{N \to \infty} P\left(\bigcup_{i=1}^{N^4} Q_i\right) = 0$$

which will follow from

$$P(Q_i) < Ke^{-\gamma N} \tag{3.4}$$

where K and γ are positive constants independent of N and i.

Let $1 \leqslant i \leqslant N^4$. Choose $k < 1/\alpha_r$. Then write

$$\begin{aligned} P(Q_i) < P(Q_i &\cap \{\forall s \in [\tau_i, \tau_i + kN], l_s^{N \times \tau_i} > (1-\beta)\, N\} \\ &\cap \{0 \times [\tau_i, \tau_i + kN] \leftrightarrow [N, \infty) \times (\tau_i + kN)\}) \\ &+ P(\exists s \in [\tau_i, \tau_i + KN], l_s^{N \times \tau_i} \leqslant (1-\beta)\, N\} \\ &+ P(0 \times [\tau_i, \tau_i + kN] \not\leftrightarrow [N, \infty) \times (\tau_i + kN)\}) \end{aligned}$$

For the last addend, the Markov property, self-duality, and an edge speed result yield

$$P(0 \times [\tau_i, \tau_i + kN] \not\leftrightarrow [N, \infty) \times (\tau_i + kN)) \leqslant P(r_{kN}^{(-\infty, 0]} \leqslant N) < Ke^{\gamma N}$$

Another edge speed result yields exponential decay of the second addend, as the left edge speed is zero. Finally, the first addend is equal to zero. To see this, take $N^2 t \in (\tau_i, N^2 S_N]$ and $x \in (0, (1-\beta)\, N]$ and realize that

$$(N, \tau_i) \leftrightarrow (x, N^2 t)$$

$$\forall s \in [\tau_i, \tau_i + kN], \qquad l_s^{N \times \tau_i} > (1-\beta)\, N$$

and

$$0 \times [\tau_i, \tau_i + kN] \leftrightarrow [N, \infty) \times (\tau_i + kN)$$

imply that there is a path from $G_N \times 0$ to $(x, N^2 t)$—that is, $\eta^{G_N}_{N^2t}(x) = 1$. So (3.4) and hence (3.2) follow routinely. ∎

4. PROOF OF PROPOSITION 2.1

In this section, we will apply Proposition 3.1 to prove Proposition 2.1. Define

$$A^i_N = \{\, |H^i_N \cap [\varepsilon N, \varepsilon N + \sqrt{N}]| \geqslant \log N\}$$

If $\liminf_N P(A^i_N) > 0$, then Proposition 3.1 says that X^i_N on A^i_N converges in distribution to $\varepsilon + B$, where B is a Brownian motion. If we could show that

$$\lim_{N \to \infty} P\left(\bigcap_j A^j_N\right) = 1 \tag{4.1}$$

then we could finish the proof of Proposition 2.1 without difficulty. Those extra routine technicalities will be omitted here; the proof will be completed by showing (4.1).

Proof of (4.1). First (suppressing N whenever it is convenient), let

$$D^i = \{X^i(T^i) = 0\}$$

We know that

$$P\left(\bigcap_j A^j_N\right) = P((D^1)^c) + P(D^1 \cap A^2 \cap (D^2)^c)$$
$$+ P(D^1 \cap A^2 \cap D^2 \cap A^3 \cap (D^3)^c) + \cdots$$

However, since A^i and D^i are all increasing events,

$$P(A^k \mid D^1 \cap A^2 \cap \cdots \cap D^{k-2} \cap A^{k-1} \cap D^{k-1}) = P(A^k \mid D^{k-1} \cap A^{k-1})$$
$$\geqslant P(A^k \mid D^{k-1})$$

where the inequality follows from the FKG inequality. (For the FKG inequality, see Liggett,[8] for example.) Also,

$$P(D^k \mid D^1 \cap A^2 \cap \cdots \cap D^{k-1} \cap A^k) = P(D^k \mid D^{k-1} \cap A^k)$$

Putting everything together, we have

$$P\left(\bigcap_j A^j_N\right) \geqslant \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} d^i_N a^{i+1}_N\right)(1 - d^k_N) \tag{4.2}$$

where

$$a_N^k = P(A^k \mid D^{k-1})$$

and

$$d_N^k = P(D^k \mid D^{k-1} \cap A^k)$$

Now we will prove (4.1) by showing the following two results:

$$\lim_{N \to \infty} a_N^i = 1 \tag{4.3}$$

$$\lim_{N \to \infty} d_N^i = 1 - \varepsilon \tag{4.4}$$

Indeed, the proof of (4.1) finishes easily from (4.2)–(4.4) as follows: Given $\beta > 0$ we can pick K, δ so that

$$(1-(1+\delta)(1-\varepsilon))(1-(1-\varepsilon))^{-1}(1-(1-\varepsilon)^K)(1-\delta)^{2(K-1)} > 1-\beta$$

Then, using (4.3) and (4.4), we know that for large enough N and for $i \leqslant K$, $a_N^i > 1-\delta$ and $d_N^i \in ((1-\delta)(1-\varepsilon), (1+\delta)(1-\varepsilon))$. Finally, using (4.2), we have

$$P\left(\bigcap_j A_N^j\right) \geqslant \sum_{k=1}^{K} \left(\prod_{i=1}^{k-1} d_N^i a_N^{i+1}\right)(1-d_N^k) > 1-\beta$$

We will now prove (4.3) and (4.4) together by induction on i. Clearly $a_N^1 = 1$; as for d_N^1, the proof is similar to the induction step and so we avoid writing the same proof twice. Now, assume (4.3) and (4.4) for all values less than or equal to i. We seek to show

$$P(|H^{i+1} \cap [\varepsilon N, \varepsilon N + \sqrt{N}]| \geqslant \log N \mid D^i) \to 1 \tag{4.5}$$

and

$$P(D^{i+1} \mid D^i \cap A^{i+1}) \to 1 - \varepsilon \tag{4.6}$$

To show (4.5), we first show the following two claims:

$$\lim_N P(T_N^i - T_N^{i-1} > 1/N \mid D^i) = 1 \tag{4.7}$$

and

$$\lim_N P(0 \times (N^2 T_N^i - N, N^2 T_N^i] \leftrightarrow [\varepsilon N + \sqrt{N}/2, \varepsilon N + \sqrt{N}] \times N^2 T_N^i \mid D^i) = 1 \tag{4.8}$$

To show claim (4.7), first notice that

$$\lim_N P(l_t^{[\varepsilon N, \infty) \times N^2 T_N^{i-1}} > 0, \forall t < N \mid D^{i-1}) = \lim_N P(l_t^{[\varepsilon N, \infty)} > 0, \forall t < N)$$
$$= 1 \tag{4.9}$$

However, we also have that

$$\liminf_N P(D^i \mid D^{i-1}) \geqslant 1 - \varepsilon \tag{4.10}$$

since, by induction,

$$P(D^i \mid A^i \cap D^{i-1}) \to 1 - \varepsilon$$

and

$$P(A^i \mid D^{i-1}) \to 1$$

It follows from (4.9) and (4.10) that

$$P(l_t^{[\varepsilon N, \infty) \times N^2 T_N^{i-1}} > 0, \forall t < N \mid D^i) \to 1$$

This is sufficient to justify the claim (4.7).

Now, to justify claim (4.8), let $\hat{l}$ denote the left edge of the dual process $\hat{\xi}$ and proceed as follows (using the FKG inequality for the first step):

$$\lim_N P(0 \times (N^2 T_N^i - N, N^2 T_N^i] \leftrightarrow [\varepsilon N + \sqrt{N}/2, \varepsilon N + \sqrt{N}] \times N^2 T_N^i \mid D^i)$$

is bounded below by

$$\lim_N P(0 \times (N^2 T_N^i - N, N^2 T_N^i] \leftrightarrow [\varepsilon N + \sqrt{N}/2, \varepsilon N + \sqrt{N}] \times N^2 T_N^i)$$

which is bounded below by

$$\lim_N P(\hat{l}_N^{[\varepsilon N + \sqrt{N}/2, \infty]} < 0) - P([\varepsilon N + \sqrt{N}/2, \varepsilon N + \sqrt{N}] \text{ dies out in } \hat{\xi})$$

But

$$\lim_N P(\hat{l}_N^{[\varepsilon N + \sqrt{N}/2, \infty]} < 0) = \lim_N P(r_N^{(-\infty, 0]} > \varepsilon N + \sqrt{N}/2)$$

(for small enough ε) and (by Theorem 13 of Schonmann[10])

$$\lim_N P([\varepsilon N+\sqrt{N}/2, \varepsilon N+\sqrt{N}] \text{ dies out in } \hat{\xi})=0$$

So, we have justified the claim (4.8).

Now, let

$$B=\{0\times(N^2T_N^{i-1}, N^2T_N^i] \leftrightarrow [\varepsilon N+\sqrt{N}/2, \varepsilon N+\sqrt{N}]\times N^2T_N^i\}$$

The importance of the claims (4.7) and (4.8) is that they easily imply

$$\lim_N P(B \mid D^i)=1$$

So, we now may condition on the event B whenever we condition on the event D^i and take the limit in N.

Next, let M be the greatest integer less than $\sqrt{N}/\log N$ and let $I_1,..., I_{\log N}$ be disjoint subintervals of $[\varepsilon N, \varepsilon N+\sqrt{N}/2)$ of length M. Of course, we are currently trying to show (4.5); clearly, it suffices to show that

$$\lim_N P(\forall j, I_j \cap H^{i+1} \neq \varnothing \mid B\cap D^i)=1$$

Now, on the event $B\cap D^i$, the existence of a path from $\mathbf{Z}\times N^2T_N^{i-1}$ to $x\times N^2T_N^i$ (where $x\in I_j$) implies the existence of a path from $(H_N^i \cap [\varepsilon N, N))\times N^2T_N^{i-1}$ to $x\times N^2T_N^i$. Indeed, whichever point in $(H_N^i\cap [\varepsilon N, N))\times N^2T_N^{i-1}$ connects (via η) to $0\times N^2T_N^i$ would also connect to $x\times N^2T_N^i$. So,

$$\begin{aligned}
&\lim_N P(\forall j, I_j\cap H^{i+1}\neq\varnothing \mid B\cap D^i)\\
&\geqslant \lim_N P(\forall j, \mathbf{Z}\times N^2T_N^{i-1} \leftrightarrow I_j\times N^2T_N^i \mid B\cap D^i)\\
&\geqslant \lim_N P(\forall j, \mathbf{Z}\times N^2T_N^{i-1} \leftrightarrow I_j\times N^2T_N^i)\\
&\geqslant \lim_N \prod_{j=1}^{\log N} P(\mathbf{Z}\times N^2T_N^{i-1} \leftrightarrow I_j\times N^2T_N^i)\\
&\geqslant \lim_N \prod_{j=1}^{\log N} P(I_j \text{ survives in } \hat{\xi})\\
&\geqslant \lim_N (1-Ce^{-\gamma M})^{\log N}\\
&=1
\end{aligned}$$

where the second and third inequalities come from the FKG inequality and the fifth inequality comes from Theorem 13 of Schonmann.[10]

So we have now shown (4.5). As for the proof of (4.6), first notice that, by induction,

$$\begin{aligned}\liminf_N P(D^i \cap A^{i+1}) &\geqslant \liminf_N d^1 a^2 d^2 a^3 \cdots d^i a^{i+1}\\ &= (1-\varepsilon)^i\\ &> 0\end{aligned}$$

We now look to Proposition 3.1, which the above calculation allows us to apply. Given D^i—that is, given $X^i(T^i)=0$—we know that X^{i+1} (on $[0, T^{i+1}-T^i]$) is distributed as the process Z in Proposition 3.1. Thus, using that proposition and noticing that $1-\varepsilon$ is the probability that a Brownian motion hits $-\varepsilon$ before $1-\varepsilon$, we easily obtain (4.6), the result we seek. ▌

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